Noise vs Computational unpredictability in dynamics

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July 10, 2013

Predicting Natural Phenomena: can we compute the future?

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More generally

Given an evolving system, can we compute its long term prospects?

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  but... is this a prevalent situation? does it occur with positive probability? does it persist after small perturbations?

Dynamical systems and Natural Phenomena: mathematical models

A dynamical system is a space of states X together with a map $T : X \to X$.

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- The "frontiers" between regions (basins) are invariant "repellers" supporting other invariant measures.

Some examples

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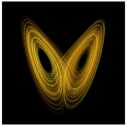
- Lorenz equations



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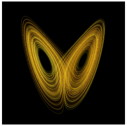
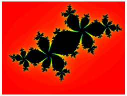


- Polynomials on the complex plane (Julia sets: repellers)



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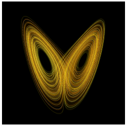

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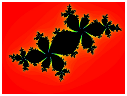

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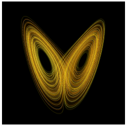

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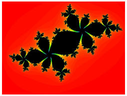

- symbolic systems: cellular automata, subshifts
- piece-wise linear transformations
  - Neural networks – agent systems (high dimensional)
  - Billiards, ray-tracing (low-dimensional)

Computation and Dynamical Systems: interactions

Dynamical systems as computing machines

How much computational power does a dynamical system have?

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- Tilings of the plane $\equiv$ full-turing power (Berger, Robinson)

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What dynamical features can be computed ?

Positive results:

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- Invariant measures are computable for:
  - Piece-wise expanding maps and hyperbolic systems (Galatolo, Hoyrup, R.)
  - Harmonic measure on Julia sets (Binder, Braverman, Yampolsky, R.)

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- For ergodic systems there exists computable *generic* points (Avigad, Gerhardy, Towsner, Gacs, Galatolo, Hoyrup, R.).

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- There exists uncomputable Julia sets (Braverman, Yampolsky)
- There exists computable systems *without* computable invariant measures (Galatolo, Hoyrup, R.).

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Complexity in dynamical systems

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- more examples in the next talk...
- ...

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Conjecture

Uncomputable/intractable phenomena cannot occur robustly in "reasonably constrained" systems.

Our contribution

Uncomputablity is not robust

Given a system T, we consider a small random perturbation T_ε of it.

Idea: x goes to $T(x)$ and then disperses randomly with distribution $p_{\varepsilon,T(x)}$. Where $p_{\varepsilon,x} \to \delta_x$ as $\varepsilon \to 0$.

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Theorem A.(Braverman, Grigo, R.) Let T be a computable system over a compact subset X of $\mathbb{R}^d$. *Assume* $p_{\varepsilon,T(x)}$ *is uniform on the* ε*-ball around* $T(x)$. *Then, for almost every* $\varepsilon > 0$, *the ergodic measures of the perturbed system* T_ε *are all computable.*

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- Intuitively, this says that the uncomputable phenomena is broken by the noise.

Our contribution

Intractability is not robust

Theorem B. (Braverman, Grigo, R.) *Suppose the perturbed system T_ε is uniquely ergodic and the function T is poly-time computable. Then there exists an algorithm $\mathcal{A}$ that computes μ with precision α in time $O_{T,\varepsilon}(poly(\frac{1}{\alpha}))$.*

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Remarks:

- The upper bound is exponential in the number of precision bits.
- The algorithm can be implemented using $poly(log(\frac{1}{\alpha}))$ space.

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Theorem C. (Braverman, Grigo, R.) *If the noise "is nice" (is not a source of additional complexity), then the computation of* μ *at precision* $\alpha < O(\varepsilon)$ *requires time* $O_{T,\varepsilon}(poly(\log \frac{1}{\alpha}))$.

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Remark:

- Intuition: at scales below the noise level, the "computationally simple" behavior takes over.

Some details about the results

Statistical behavior: Invariant and ergodic masures

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$$\mu(T^{-1}E) = \mu(E) \qquad \text{for every Borel set } E.$$

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Invariant measures correspond to *equilibrium states* of the system. The **ergodic** measures are the ones that can not be *decomposed*:

$$\text{For every invariant set } E, \text{ either } \mu(E) = 1 \quad \textit{or} \quad \mu(E) = 1.$$

Statistical behavior

Small random perturbations

Here X is a space on which Lebesgue measure can be defined. Consider a family $\{p_x^\varepsilon\}_{x \in X} \in M(X)$ (a probability kernel) such that

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Definition

A **random perturbation of** T, T_ε is a Markov Chain X_n, $n = 0, 1, 2, \ldots$ with transition probabilities $P(A|x) = p^\varepsilon_{T(x)}(A)$. Given $\mu \in M(X)$, the **push forward** of μ under T_ε is defined by $(T_\varepsilon \mu)(A) = \int_X P(A|x)\, d\mu$.

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Definition

A probability measure μ on X is called an **invariant measure of the random perturbation** T_ε **of** T if $T_\varepsilon \mu = \mu$.

The space of measures

Let M_{inv} denote the space of invariant probability measures.

- M_{inv} is a compact, convex, non empty set,
- The extremal points are the **ergodic** measures,
- if M_{inv} contains just one measure, then the system is called **uniquely ergodic**.

Which invariant measures are computable ?

Computability of probability measures

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Proposition

The triple $(M(X), \mathcal{D}, \rho)$ *is a computable metric space.*

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... so we have *a* notion of computable measure to work with.

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- The pushforward (or transition) operator $\mathcal{P} : \mu \to T_\varepsilon \mu$ is computable.
- If X is effectively compact, so is M_{inv}.
- It follows that uniquely ergodic systems have a computable invariant measure.

Main proof ideas

Proof of Theorem A

Theorem A. If $p_{\varepsilon,T(x)}$ *is uniform on the* ε*-ball around* $T(x)$. *Then, for almost every* $\varepsilon > 0$, *the (finitely many) ergodic measures of the perturbed system* T_ε *are all computable.*

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- T_ε can have at most finitely many ergodic measures.

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  (ii) for every $x \in A_i$, $\mu_x = \mu_i$, where μ_x is the limiting distribution of T_ε starting at x.
- We can "explore" the space to algorithmically find regions A_i like above.
- Restricted to each region, T_ε is uniquely ergodic. Computability of each measure now follows from compactness.

Main proof ideas

Proof of Theorem B

Theorem B. *Suppose the perturbed system T_ε is uniquely ergodic and the function T is polynomial-time computable. Then there exists an algorithm A that computes μ with precision α in time $O_{T,\varepsilon}(poly(\frac{1}{\alpha}))$.*

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Remarks:

- The upper bound is exponential in the number of precision bits.
- Upon input α, the algorithm outputs a list $\{w_\mathfrak{a}\}_{\mathfrak{a}\in\zeta}$ of $poly(1/\alpha)$ dyadic numbers representing the piece-wise constant function

$$\mathcal{A}(\alpha) = \sum_{\mathfrak{a}\in\zeta} w_\mathfrak{a} \mathbf{1}\{x \in \mathfrak{a}\}$$

where P is a regular-size partition with $poly(1/\alpha)$ pieces.

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Proof of Theorem B

Idea: exploit the mixing properties $\mathcal{P}$.

- Since $\mathcal{P}$ may not have a spectral gap, we construct a related transition operator $\overline{\mathcal{P}}$ that has the same invariant measure as $\mathcal{P}$ while having a a spectral gap.

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  iii) the density associated to ψ is L^1-close to the stationary distribution of $\mathcal{P}$.
- Q corresponds (roughly) to a piece-wise constant approximation of $\mathcal{P}$ on a finite partition ζ.
- Computing μ here means to have the vector ψ.

Main proof ideas

Proof of Theorem C

Theorem C. *Suppose the noise* $p^{\varepsilon}_{T(x)}(\cdot)$ *is "nice". Then the computation of* μ *at precision* $\delta < O(\varepsilon)$ *requires time* $O_{T,\varepsilon}(poly(\log \frac{1}{\delta}))$.

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- Here we actually prove that μ has a poly-time computable analytic density. And therefore $\mu[0, x]$ is poly-time computable.
- The noise kernel $p^{\varepsilon}(y, x)$ is "nice" if there exists constants $C > 0$ and $\gamma > 0$ such that

$$|\partial_2^k p_{\varepsilon}(y, x)| \leq C\, k!\, e^{\gamma k} \qquad \text{for all } k \in \mathbb{N} \text{ and all } x, y \in X.$$

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- In particular, $P\nu(dx)$ has a density for any probability measure ν.

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How to get rid of this exponential approximation?

Main proof ideas

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Solution:

- We use a fixed partition ζ that depends only on the noise ($\operatorname{diam}\zeta < \frac{1}{e^{\gamma}}$).

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- The regularity of the kernel implies the regularity of $\mathcal{P}\rho$, for any initial density ρ.
- This provides an "infinite" matrix representation for $\mathcal{P}$, organized in a fixed number of blocks.

Main proof ideas

Proof of Theorem C

- We now can *truncate* the series representations and get a finite matrix P_N, corresponding to a finite approximation of $\mathcal{P}$.

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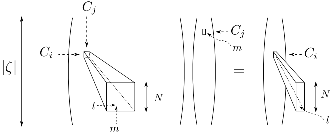

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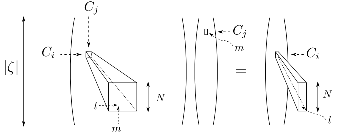

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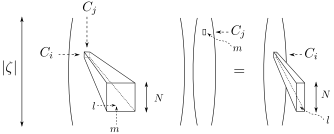

Figure: Graphical representation of the equation $P_N \, \rho_N^{(t)} = \rho_N^{(t+1)}$.

- The size of P_N depends **linearly** on the number n of precision bits !
- The invariant density π is computed by iterating $P_N \, \rho_N^{(t)}$ of any initial density ρ sufficiently many times (also linear in n) and then use the resulting vector and Taylor formula to compute $\pi(x)$.

Further work

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- Hard to formalize.

THANKS !